ON THE COMPUTATION OF STATISTICAL CHARACTERISTICS OF PRESSURE PULSATIONS ON THE SURFACE OF A PLATE BENEATH A TURBULENT BOUNDARY LAYER

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A method for the computation of statistical characteristics of pressure pulsations directly at a plate surface, based on the analysis of the Navier-Stokes equations is proposed. This permits in the case of flows past a plate to derive the unique relationship between the second order moment of pressure pulsation at the surface and the second order moment of the pulsation velocity longitudinal component in the immediate proximity of the plate surface. Information about the velocity field required in this case is considerably smaller than that necessary in the conventional approach used by Kraichnan [1] and Lilley [2].

1. We shall consider the turbulent boundary layer of an incompressible fluid along an infinite flat plate defined by Eq. $x_2 = 0$. Coordinates x_1 and x_3 lie in the plate surface plane. In this case the flow is subject to the Navier-Stokes equations and to the incompressibility equation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu\nabla^2 v_i, \qquad \frac{\partial v_j}{\partial x_j} = 0 \quad (i = 1, 2, 3) \tag{1.1}$$

Here, t is the time, p the pressure, ρ_0 the fluid density, ν the viscosity, and v_i the component of velocity $\mathbf{v}$ in the direction of the x_i-axis, with $v_i = 0$ when $x_2 = 0$.

We shall examine the behavior of these equations in the close proximity of a smooth flat wall ($x_2 = 0$, $-\infty < x_1 < +\infty$, $-\infty < x_3 < +\infty$). Ladyzhenskaia in her monograph [3] and in her subsequent paper [4] had shown that there exists a solution of Eq. (1.1), and that for a sufficiently smooth surface $S\,(S \in L_2)$ the derivatives of velocity $\mathbf{v}$, including those of the second order are continuous in the bounded area, while first order derivatives of pressure p are continuous up to the boundary.

On this basis we can examine Eqs. (1.1) at the surface itself. When $x_2 \to +0$, we obtain from the first of Eqs. (1.1) three relationships which correlate pressure gradients and the second derivatives of velocities at the surface

$$\left.\frac{\partial p}{\partial x_k}\right|_{x_2=0} = \mu \left.\frac{\partial^2 v_k}{\partial x_2^2}\right|_{x_2=0} \qquad \left(\mu = \frac{\nu}{\rho_0}\right) \quad (k = 1, 2, 3) \tag{1.2}$$

We have made use here of the fact that at the stationary surface $x_2 = 0$ not only all $v_i = 0$ ($i = 1, 2, 3$), but also $\partial v_i / \partial t = 0$, and that furthermore $\partial^{k+p} v_i / \partial x_j^k \partial x_l^p = 0$ when $j, l \neq 2$ ($j = 1, 2, 3$).

Eqs. (1.2) with subscripts $k = 1$ and $k = 3$ represent a system of differential equations in second order partial derivatives of pressure which completely defines pressure variation along the plate surface. A single differential equation of the Poisson type may be readily obtained from this for the pressure distribution along the plate surface only

$$\nabla^2_{x_1x_3} p(x_1, 0, x_3, t) = \mu \left.\frac{\partial^3 v_i(x_1, x_2, x_3, t)}{\partial x_2^2 \partial x_i}\right|_{x_2=0} \qquad (i = 1, 3) \tag{1.3}$$

The linearity of the first part of Eq. (1.3) gives it a certain advantage over the equation previously used by Kraichnan [1] and Lilley [2]

$$\nabla^2_{x_1, x_2, x_3} p(x_1, x_2, x_3, t) = -\rho_0 \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \tag{1.4}$$

The derived Eq. (1.3) is not inconsistent with Eq. (1.4) as regards the pressure distribution at any point of a turbulent stream, and is a particular case of the latter.

The computation of pressure characteristics on the surface by means of Eq. (1.3) is also preferable because, first of all, it makes it possible to do with a lesser amount of information about the boundary layer velocity field, and secondly it involves the solution of a two-dimensional problem instead of having to integrate Eq. (1.4) over the whole volume.

The equation of pressure fluctuations in a turbulent boundary layer at the plate surface is readily obtained from (1.3)

$$\nabla^2 p'(x_1, 0, x_3, t) = \mu \left.\frac{\partial^3 v_i'}{\partial x_2^2 \partial x_i}\right|_{x_2=0} \qquad (i = 1, 3) \tag{1.5}$$

Eq. (1.5) is valid in the case of the basic stream being parallel to the plate (nongradient flow), as well as in the case of the free stream flowing at an angle to the plate (positive, or negative mean pressure gradient).

2. In space-time terms the Fourier-Stiltjes transform for velocity and pressure fluctuations [5], Eq. (1.5) is expressed by (2.1)

$$dp'(k_1, 0, k_3, \omega) = \frac{ik_1\mu}{k_1^2 + k_3^2} \left.\frac{\partial^2 dv_1'(k_1, x_2, k_3, \omega)}{\partial x_2^2}\right| + \frac{ik_3\mu}{k_1^2 + k_3^2} \left.\frac{\partial^2 dv_3'(k_1, x_2, k_3, \omega)}{\partial x_2^2}\right|_{x_2=0}$$

Here ω is the temporal frequency, and k (k_1, k_3) the wave vector in the plate plane.

We shall take into account the relationship between the third derivatives of velocity fluctuations (v_1' and v_3') which is easily obtained from (1.2)

$$\left.\frac{\partial^3 v_1'}{\partial x_2^2 \partial x_3}\right|_{x_2=0} = \left.\frac{\partial^3 v_3'}{\partial x_2^2 \partial x_1}\right|_{x_2=0} \tag{2.2}$$

As a result Eq. (2.1) is reduced to the form as follows:

$$dp'(k_1, 0, k_3, \omega) = \frac{i\mu}{k_1} \left.\frac{\partial^2 dv'_{v_1}(k_1, x_2, x_3', k_3, \omega)}{\partial x_2^2}\right|_{x_2=0} \tag{2.3}$$

From (2.3) we establish in the usual manner the relation between the space-time spectra of the fluctuating pressure at the wall and the fluctuating velocity longitudinal component

$$E_{pp}(k_1, k_3, \omega) = \frac{\mu^2}{k_1^2} \left.\frac{\partial^4 E_{v_1v_1}(k_1, x_2, x_2', k_3, \omega)}{\partial x_2^2 \partial x_2'^2}\right|_{x_2=x_2'=0} \tag{2.4}$$

The term $E_{pp}(k_1, k_3, \omega)$ will be understood to represent the average of the ensemble of patterns of the turbulent boundary layer at the plates

$$E_{pp}(k_1, k_3, \omega) = \lim \frac{\langle dp'(k_1, 0, k_3, \omega)\, \overline{dp'(k_1, 0, k_3, \omega)}\rangle}{dk_1\, dk_3\, d\omega}$$

for $dk_1 \to 0$ we have $dk_3 \to 0$, $d\omega \to 0$.

From the space-time spectrum $E_{pp}(k_1, k_3, \omega)$ we can obtain the reciprocal spectrum $\Gamma_{pp}(\xi_1, \xi_3, \omega)$ and the pressure correlation $R_{pp}(\xi_1, \xi_3, \tau)$, if we assume that spectrum $E_{pp}(k_1, k_3, \omega)$ corresponds to a boundary layer uniform in directions parallel to the plate surface and stationary with respect to time.

Here $\xi_1 = x_1 - x_1'$, $\xi_3 = x_3 - x_3'$; $\xi(\xi_1, \xi_3)$ is the distance between points under consideration, and $\tau = t - t'$ the time difference at these points. In a developed turbulent layer at the surface of a smooth plate the conditions of uniformity and stationarity for zero mean pressure gradient are approximately satisfied.

Relationship (2.4) which is fundamental in this analysis defines the statistical characteristics *of pressure at the surface in terms of the correlation function of the* velocity longitudinal component, since

$$E_{v_1v_1}(k_1, x_2, x_2', k_3, \omega) = \tag{2.5}$$
$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{v_1v_1}(\xi_1, x_2, x_2', \xi_3, \tau) \times \exp(ik_1\xi_1 + ik_3\xi_3 + i\omega\tau)\, d\xi_1\, d\xi_3\, d\tau$$

3. Even the most detailed measurements of the correlation function of the longitudinal component *of velocity in a boundary layer carried out by* Favre [6 and 7] are insufficient for the construction of a correlation function at various distances from, but in close proximity of a wall. We shall, first of all express $R_{v_1v_1}(\xi_1, x_2, x_2', \xi_3, \tau)$ in terms of intensity of the velocity longitudinal component at various distances from the plate $\langle v_1'^2\rangle^{1/2}$ and of the *dimensionless correlation coefficient* $r_{v_1v_1}(\xi_1, x_2, x_2', \xi_3, \tau)$. The *corresponding relation* between the spectra of these parameters is of the form

$$E_{v_1v_1}(k_1, x_2, x_2', k_3, \omega) = \langle v_1'^2(x_2)\rangle^{1/2} \langle v_1'^2(x_2')\rangle^{1/2} e_{v_1v_1}(k_1, x_2, x_2', k_3, \omega) \tag{3.1}$$

Here $e_{v_1v_1}$ is the spectrum of the correlation coefficient $r_{v_1v_1}$.

As long as the boundary layer under consideration is uniform in planes parallel to the plate surface, as it is assumed to be in the following, the intensity of the velocity longitudinal component will remain a function of distance from the surface only. Experimental data of Klebanov and Laufer on measurements of the longitudinal component in the wall neighborhood processed by Monin and Iaglom [8] yield for the wall neighborhood

$$\langle v_1'^2(x_2)\rangle^{1/2} \approx a\frac{v^{*2}x_2}{\nu} \tag{3.2}$$

Here $a \approx 0.3$, and v^* is the dynamic velocity. Experimental data [6 and 7] do not allow to predict variations of $e_{v_1v_1}(k_1, x_2, x_2', k_3, \omega)$ in the immediate vicinity of a wall. It is not difficult to ascertain with the aid of (2.4) that the presentation of $e_{v_1v_1}(k_1, x_2, x_2', k_3, \omega)$ in the form $e_{v_1v_1}(k_1, x_2, x_2', k_3, \omega) = r_{v_1v_1}(x_2, x_2')\, e_{v_1v_1}(k_1, k_3, \omega)$ is not possible because of the boundedness of the pressure real spectrum $e_{pp}(k_1, k_3, \omega)$ when $k_1 = 0$.

One of the possible and probable approximations of the velocity longitudinal component spectrum which would satisfy the condition of the pressure boundedness at $(\cdot)\, k_1 = 0$ is a function of the form (used by us in the following):

$$e_{v_1v_1}(k_1, x_2, x_2', k_3, \omega) = \exp[-\gamma k_1 |x_2 - x_2'|]\, e_{v_1v_1}(k_1, +0, +0, k_3, \omega) \tag{3.3}$$

Function $e_{v_1v_1}(k_1 + 0, 0, k_3, \omega)$ will be understood to be the space-time spectrum of the velocity longitudinal component at very small distances from the plate surface.

Favre's experimental data [6 and 7] indicate that the second order moment of the longitudinal component satisfies the following fundamental requirements. First, the double correlation $r_{v_1v_1}(\xi_1, \tau)$ attains its maximum value at a certain optimal time lag $T \approx \xi_1/V$ where V is the convection velocity of transport of inhomogeneities in the turbulent boundary layer by means of the average motion ($V \approx 0.8\, V_0$). This correlation coefficient is roughly symmetric with respect to the optimal time lag. Curves corresponding to the maximum of the correlation coefficient $r_{v_1v_1} = r_{t_1v_1}(\xi_1) = (\xi_1, x_2, x_2, \tau)$ at various distances from the plate surface are shown on Fig. 1. Curves *4* have been plotted for $x_2 \gg \delta$, curve *3* for $x_2 = 0.24\delta$, curve *2* for $x_2 = 0.06\delta$, and curve *1* for $\tau = 0$. Second, the autocorrelation coefficient $r_{v_1v_1}(\tau)$ (Fig. 2), and the coefficient $r_{v_1v_1}(\xi_1)$ of longitudinal correlation (Fig. 3) are of a nearly exponential form. Curve *1* on Fig. 3 corresponds to the longitudinal correlation for $x_2 = 0.14\delta$, and curve *2* for $x_2 = 0.29\delta$.

We shall consider two models of correlation function

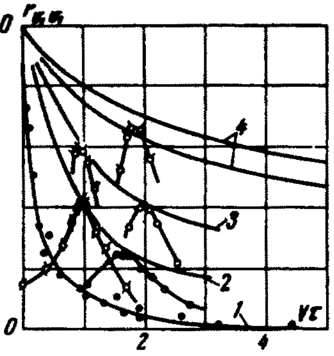

Fig. 1

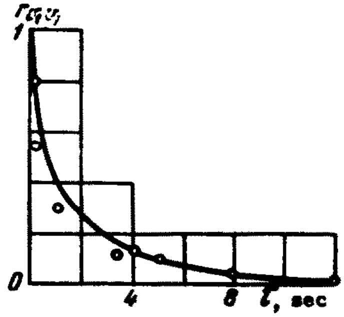

Fig. 2

tion $R_{v_1v_1}(\xi_1, \xi_3, \tau)$ which conform to these experimental relationships:

a) the Taylor frozen turbulence model with a purely convective transport of inhomogeneities by the average motion

$$r_{v_1v_1}(\xi_1, \xi_3, \tau) = \exp[-\alpha|\xi_1 - V\tau|] \exp[-g|\xi_3|] \quad (3.4)$$

b) a somewhat more complicated model in which account is taken of the turbulent vorticity degeneration in the process of their transport in the direction of the average motion. The corresponding correlation coefficient is of the form

$$_{v_1v_1}(\xi_1, \xi_3, \tau) = \exp[-\alpha|\xi_1 - V\tau| - \beta|\xi_1|] \exp[-g|\xi_3|] \quad (3.5)$$

In the following we shall use the latter form of the longitudinal component correlation function. Parameters α and β determined from the comparison of (3.5) with curves plotted in Figs. 1, 2 and 3 are $\alpha = 2.0/\delta$, $\beta = 1.4/\delta$. Here δ is the boundary layer thickness. It is not possible to determine the value of coefficient g because of the lack at present of experimental data on the correlation of the velocity longitudinal component in a direction parallel to the plate surface and normal to the direction of the main stream. Coefficient γ is selected from the comparison of $r_{v_1v_1}(x_2 - x_2')$ with the correlation coefficient $r_{v_1v_1}(x_2, x_2 - x_2')$ calculated from Favre's experimental data [6] (Fig. 4). Curve *1* on Fig. 4 corresponds to coefficient $r_{v_1v_1}(x_2, x_2 - x_2')$ for distance $x_2 = 0.062\delta$, curve *2* for $x_2 = 0.25\delta$ and curve *3* for $x_2 = 0.75\delta$. Analysis of curve behavior at the smallest distances of the reference point x_2 from the plate surface in the neighborhood of $x_2 - x_2' = 0$ yields the value $\gamma = 4.9$.

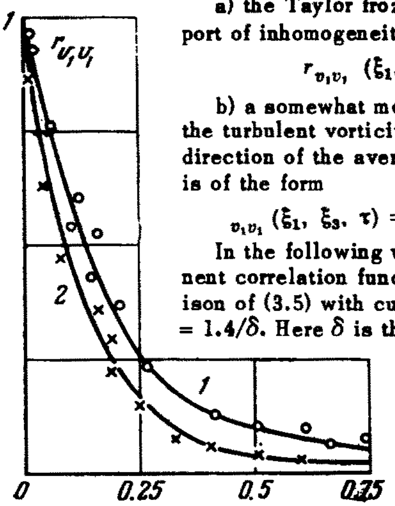

Fig. 3

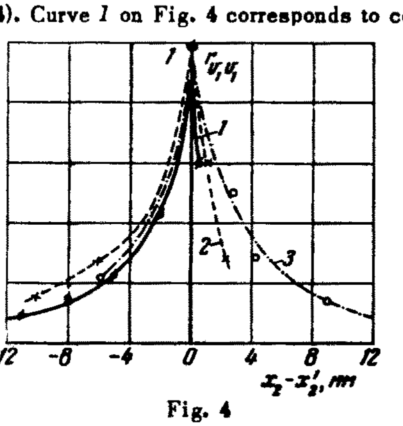

Fig. 4

4. In order to obtain the pressure spectrum at the plate surface we substitute Expressions (3.1) and (3.3) into the right-hand side of solution (2.4)

$$E_{pp}(k_1, k_3, \omega) = \quad (4.1)$$
$$= 4a^2\gamma^2\rho_0^2\, v^{*4}\, e_{v_1v_1}(k_1, +0, +0, k_3, \omega)$$

Since $\tau_w = \rho_0 v^{*2}$, hence we rewrite (4.1) in the form

$$E_{pp}(k_1, k_3, \omega) = 4a^2\gamma^2\tau_w^2\, e_{v_1v_1}(k_1, +0, +0, k_3, \omega) \tag{4.2}$$

Passing in (4.2) from spectra to correlation functions, we obtain the reciprocal pressure spectrum

$$\Gamma_{pp}(\xi_1, \xi_3, \omega) = 4a^2\gamma^2\tau_w^2\ \Gamma_{v_1v_1}(\xi_1 + 0, +0, \xi_3, \omega) \tag{4.3}$$

It follows from (4.3) that for the mean square value of pressure at the wall the following relationship is valid

$$\langle p'^2\rangle = 4a^2\gamma^2\tau_w^2, \quad \langle p'^2\rangle^{1/2} = 2a\gamma\tau_w \tag{4.4}$$

Substituting into (4.4) the previously found values of a and γ, we obtain $\langle p'^2\rangle^{1/2} = 2.94\tau_w$. This result coincides with Kraichnan's conclusions [1] as to the proportionality of the mean square value of friction stress at the wall, and with Lilley's estimates of the value of the proportionality coefficient [2]

$$1.7 \leqslant \langle p'^2\rangle^{1/2}/\tau_w \leqslant 3.1$$

Eq. (4.4) indicates moreover that the proportionality coefficient is determined by the angle between the intensity profile of the pulsating velocity longitudinal component and the plate plane, and by the radii of correlation of that same velocity component.

Data on the mean square value of pressure pulsations at a plate surface obtained by Willmarth and Wooldridge [9] yield 2.7 and 2.2 as the value of this coefficient in two different conditions.

Representing the friction stress at the wall in terms of the velocity head and of the Reynolds number in the width of momentum loss [10], we obtain for $\langle p'^2\rangle^{1/2}$ the following expression

$$\langle p^2\rangle^{1/2} = 0.0131\ \ a\gamma\rho_0\ V_0^2\ R_{\delta^{**}}^{-1/6} \tag{4.5}$$

It follows from (4.5) that the mean square value of pressure is proportional to the free stream velocity with an exponent slightly smaller than two (11/6).

In order to determine from Formula (4.2) the space time pressure spectrum $E_{pp}(k_1, k_3, \omega)$ it is necessary to obtain $e_{v_1v_1}(k_1, +0, +0, k_3, \omega)$ from the correlation function $r_{v_1v_1}(\xi_1, \xi_3, \tau)$. We then have

$$E_{pp}(k_1, k_3, \omega) = \frac{32a^2\gamma^2\alpha\beta g V\tau_w^2}{(\alpha^2V^2 + \omega^2)\,[\beta^2 + (k_1 - \omega/V)^2]\,(g^2 + k_3^2)} \tag{4.6}$$

The form of the reciprocal spectrum of pressure at the plate surface which follows from (4.6) is

$$\Gamma_{pp}(\xi_1, \xi_3, \omega) = \frac{4a^2\gamma^2\alpha V\tau_w^2}{\pi(\alpha^2V^2 + \omega^2)}\exp\left[\frac{i\omega\xi_1}{V} - \beta\,|\,\xi_1\,| - g\,|\,\xi_3\,|\right] \tag{4.7}$$

In the case of coincidence of the two points $\xi_1 = \xi_3 = 0$, Eq. (4.7) becomes the power spectrum of pulsating pressure

$$\Gamma_{pp}(\omega) = \frac{4a^2\gamma^2\alpha V\tau_w^2}{\pi(\alpha^2V^2 + \omega^2)} \tag{4.8}$$

Results of direct measurements of the power spectrum of pressure at a plate obtained by Willmarth and Wooldridge [9] are shown on Fig. 5. The curves are shown in terms of the dimensionless frequency $\Omega = \omega\delta^*/V_0$ (δ^* is the width of displacement), and are normalized by magnitude $\frac{1}{4}\rho_0^2 V_0^3\delta^*$. A similar operation performed on Expression (4.8) yields the dimensionless power spectrum Γ_{pp}^* in the form as follows:

$$\Gamma_{pp}^* = \frac{\Gamma_{pp}(\Omega)}{\frac{1}{4}\rho_0^2V_0^3\delta^*} = \frac{16a^2\gamma^2\alpha V\tau_w^2}{\pi\rho_0^2V_0^3\delta^*[(0.1\alpha\delta)^2 + \Omega^2]} \tag{4.9}$$

The analytical form of (4.9) indicates the presence of a horizontal stretch at low frequencies when $\Omega \ll 0.1\,\alpha\delta$, and a drop proportional to the square of frequency when $\Omega \gg 0.1\,\alpha\delta$, while the limit frequency of the drop is determined by the radius of component v_1' longitudinal correlation. This coincides qualitatively with the behavior of curves on Fig. 5, while the apparent discrepancy in the magnitude of the limit frequency is explained by the inexact form of the correlation function of the fluctuating velocity longitudinal component.

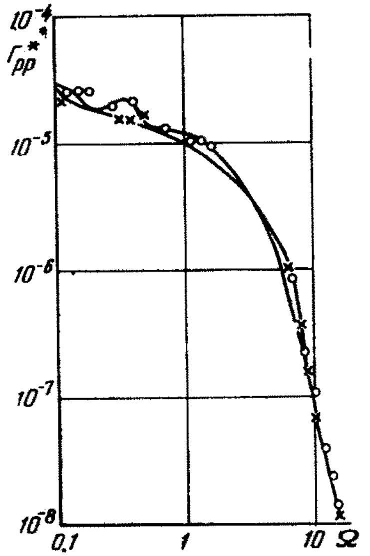

Fig. 5

The dimensionless reciprocal pressure spectrum

$$\frac{\Gamma_{pp}(\xi_1, \xi_3, \omega)}{\Gamma_{pp}(\omega)} = \exp\frac{i\omega\xi_1}{V}\exp[-\beta|\xi_1| - g|\xi_3|] \tag{4.10}$$

indicates that in the direction of the average motion the flow is unfrozen. Experimental determination of the dimensionless reciprocal spectrum by Harrison [11], and Willmarth and Wooldridge [9] indicate a somewhat different pattern of turbulence degeneration with a dimensionless reciprocal spectrum

$$\frac{\Gamma_{pp}(\xi_1, \omega)}{\Gamma_{pp}(\omega)} = \exp\frac{i\omega\xi_1}{V}\exp\left[-b\frac{\omega|\xi_1|}{V}\right]$$

but different authors do not appear to agree on the value of parameter b.

The derived pressure pulsation statistical characteristics (4.9) and (4.10) do not completely coincide with results of measurements. It may be noted, however, than none of the previous investigators had theoretically computed such characteristics (with respect to velocity fields). The complexity of Eq, (1.4) apparently precludes the derivation of results in an analytical form. With the use of Eq. (1.5) a refinement of the pressure characteristic is, however, possible, if results of precise measurements of second order moments of the longitudinal components of pulsating velocity in the immediate vicinity of a surface are made available.

It should be noted that a similar approach may be used for the determination of pressure fluctuation along a curvilinear surface in an incompressible fluid, as well as for the analysis of pressure fluctuations in turbulent boundary layers of compressible fluids.

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STABILITY OF FLOWS OF A WEAKLY COMPRESSIBLE FLUID IN A PLANE PIPE OF LARGE, BUT FINITE LENGTH

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Investigation of the stability of fluid flows in plane pipes [1] is usually associated with the investigation of the behavior, in time, of an infinite periodic wave of the form $\phi(y) \exp i(kx - \omega t)$ where k is real. The relation between ω and k is found from the condition of existence of a nontrivial solution of a boundary value problem for $\phi(y)$ and is defined by a multivalued analytic function $k(\omega)$. It was shown in [1 and 2] that the function $k(\omega)$ has only one branch $k_1(\omega)$ giving real values of k when $\mathrm{Im}\,\omega > 0$. This branch corresponds to the perturbations propagating downstream. Earlier [3] the author computed the function $k_1(\omega)$ for real ω for the case of flows of an incompressible fluid at large Reynolds' numbers. It is easily seen that the behavior of $k_1(\omega)$ will not be greatly altered when the fluid is compressible, provided that its compressibility is sufficiently small.

The condition of instability of the flow in a pipe of large but finite length, can be reduced to the fact [3 and 4] that Eq.

$$\mathrm{Im}\,[k_1(\omega) - k_a(\omega)] = 0 \tag{1}$$

has solutions ω when $\mathrm{Im}\,\omega > 0$. The expression $k_a(\omega)$ in (1) will, for the time being, denote the branch of $k(\omega)$ defining the wave number of some perturbation propagating upstream. We shall show that in the case of weakly compressible flows with high Reynolds numbers the above condition of instability holds, provided that the branch corresponding to acoustic oscillations propagating upstream is taken as $k_a(\omega)$.

If, either the fluid is compressible or the pipe walls are elastic, then acoustic or Zhukovskii waves may be set up and propagate along it. Their wavelength will, for the given frequency, be inversely proportional to the compressibility of the fluid and the walls. When the wavelength becomes large, we can neglect the transverse velocity and pressure gradient components. Excess pressure at some cross section will be proportional to the excess of mass per unit length of the pipe, so that

$$i\omega p = ik_a \rho_0 a^2 \frac{1}{2} \int_{-1}^{1} u\, dy \tag{2}$$

where k_a and ω are the wave number and frequency of the given wave, ρ_0 is the density of the fluid, a is the velocity of propagation of the perturbations and u is the longitudinal component of the velocity perturbation. In deriving (2), we have assumed that $\omega / k_a \gg u$.

Function $u(y)$ satisfies Eq.